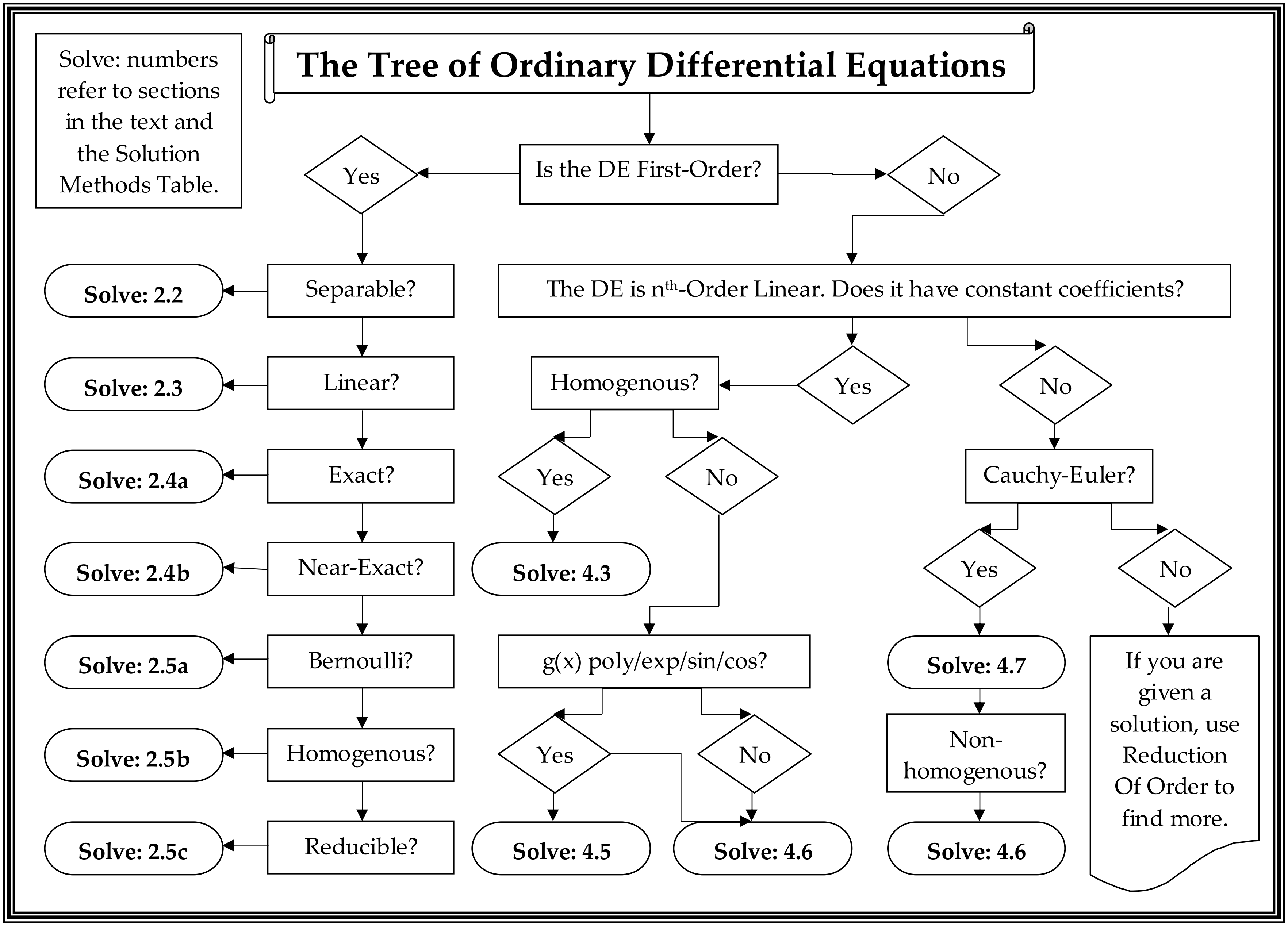

The Tree of Ordinary Differential Equations
Solve: numbers refer to sections in the text and the Solution Methods Table.
Is the DE First-Order?
Yes
No
Separable?
Solve: 2.2
Linear?
Solve: 2.3
Exact?
Solve: 2.4a
Near-Exact?
Solve: 2.4b
Bernoulli?
Solve: 2.5a
Homogenous?
Solve: 2.5b
Reducible?
Solve: 2.5c
The DE is nth-Order Linear. Does it have constant coefficients?
Yes
No
Homogenous?
Yes
No
Solve: 4.3
g(x) poly/exp/sin/cos?
Yes
No
Solve: 4.5
Solve: 4.6
Cauchy-Euler?
Yes
No
Solve: 4.7
Non-homogenous?
Solve: 4.6
If you are given a solution, use Reduction Of Order to find more.

Table of Ordinary Differential Equation Solution Methods

Type	Ordinary Differential Equations of the Form:	Method of Solution
First-Order Separable (2.2)	$\frac{dy}{dx} = g(x)h(y)$	(1) Rewrite as $\frac{1}{h(y)}dy = g(x)dx$. (2) Integrate both sides. Note: Check for singular solutions at $h(y) = 0$ which may have been lost.
First-Order Linear (2.3)	$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$	(1) Rewrite as $\frac{dy}{dx} + \frac{a_0(x)}{a_1(x)}y = \frac{g(x)}{a_1(x)}$ (standard form) (2) Determine an interval I on which both $\frac{a_0(x)}{a_1(x)}$ and $\frac{g(x)}{a_1(x)}$ are continuous. (3) Determine the integrating factor $\mu(x) = e^{\int \frac{a_0(x)}{a_1(x)}dx}$. (4) Multiply both sides of the standard-form equation by $\mu(x)$, which simplifies to $\frac{d}{dx}[\mu(x)y] = \frac{\mu(x)g(x)}{a_1(x)}$. (5) Integrate both sides of the simplified equation. Note: if $g(x) = 0$, then the equation is said to be homogenous and always has the trivial solution $y = 0$.
First-Order Exact (2.4a)	$M(x,y)dx + N(x,y)dy = 0$ (Alternatively $M(x,y) + N(x,y)\frac{dy}{dx} = 0$) And $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$	(1) Verify that $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. (2) Integrate $M(x,y)$ with respect to x (treating y as a constant). (3) Integrate $N(x,y)$ with respect to y (treating x as a constant). (4) The general solution is the sum of the results of (1) and (2), plus an arbitrary constant c.

First-Order Near-Exact (2.4b)	$M(x,y)dx + N(x,y)dy = 0$ Where $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ But $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \varphi(x)$ Or $\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \varphi(y)$	(1) First, verify whether $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \varphi(x)$ (i.e. a function of x alone) or $\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \varphi(y)$ (i.e. a function of y alone). It may be that both cases work; if so, choose the simpler function. (2) Depending on the result of (1), calculate the integrating factor, either $\mu(x) = e^{\int \varphi(x)dx}$ or $\mu(y) = e^{\int \varphi(y)dy}$ (3) Multiply $M(x,y)$ and $N(x,y)$ by the integrating factor (either $\mu(x)$ or $\mu(y)$, depending on the result of (1)). (4) Proceed as with a First-Order Exact.
First-Order Homogenous (2.5a)	$M(x,y)dx + N(x,y)dy = 0$ Where $M(x,y)$ and $N(x,y)$ are homogenous functions of the same degree, i.e. $M(tx,ty)dx + N(tx,ty)dy = t^{\alpha}[M(x,y)dx + N(x,y)dy]$ for some $\alpha \in \mathbf{R}$	(1) Verify that $M(x,y)$ and $N(x,y)$ are homogenous functions of the same degree, i.e. $M(tx,ty)dx + N(tx,ty)dy = t^{\alpha}[M(x,y)dx + N(x,y)dy]$ for some $\alpha \in \mathbf{R}$ (2) Make substitutions: a. If $M(x,y)$ is simpler than $N(x,y)$, make the substitutions $y = ux$ and $dy = xdu + udx$ b. If $N(x,y)$ is simpler than $M(x,y)$, make the substitutions $x = vy$ and $dx = ydv + vdx$ (3) Verify that the equation is now First-Order Separable (rearranging if necessary) and solve. (4) Undo the substitutions made in (2).
(First-Order) Bernoulli (2.5b)	$\frac{dy}{dx} + P(x)y = f(x)y^n$ Where $n \in \mathbf{R}$	(1) Make the substitutions $u = y^{(1-n)}$ and $du = (1-n)y^{-n}dy$ (2) Verify that the equation is now First-Order Linear (rearranging if necessary) and solve. (3) Undo the substitutions made in (1).

Reduction (2.5c)	$\frac{dy}{dx} = F(Ax + By + C)$ where $B \neq 0$ (may not have exactly this form, sometimes difficult to spot)	(1) Make the substitution $u = Ax + By + C$ and $\frac{du}{dx} = \frac{d}{dx}[Ax + By + C]$ (noting that y will become $\frac{dy}{dx}$ which will need to be solved for in order to make the substitution). (2) Verify that the equation is now First-Order Separable (rearranging if necessary) and solve. (3) Undo the substitution made in (1)
n^th^-Order Linear Homogenous (4.3)	$a_n y^{(n)} + a_{(n-1)} y^{(n-1)} + \ldots + a_0 y = 0$ Where a_n is a constant (possibly complex) and $y^{(n)}$ is the n^th^ derivative of y.	(1) Make the substitution $y = e^{mx}$ (and $y' = me^{mx}$, etc.) (2) Factor out e^{mx} from all terms. Since $e^{mx} > 0$ for all $m \in \mathbf{R}$, it can be dropped, giving the polynomial "auxiliary equation" $a_n m^n + a_{(n-1)} m^{n-1} + \ldots + a_0 = 0$ (3) Solve the auxiliary equation through whichever means are available. Classify the roots as follows: Type I – Unique Real Roots Type II – Repeated Real Roots (i.e. those roots which bring more than one factor in the auxiliary equation to zero) Type III – Complex Roots (which always appear in conjugate pairs $\alpha + \beta i$ and $\alpha - \beta i$) (4) Transform the roots of the auxiliary equation into solutions of the DE as follows: a. Type I – Unique Real Roots: For each unique real root m of the auxiliary equation, there is a solution $c \cdot e^{mx}$ to the corresponding DE. b. Type II – Repeated Real Roots: For each real root m of the auxiliary equation repeated k times, each of the following is a solution to the corresponding DE: $c_1 \cdot x^0 \cdot e^{mx}$, $c_2 \cdot x^1 \cdot e^{mx}$, … , $c_k \cdot x^{(k-1)} \cdot e^{mx}$ (This can be proved using reduction of order.) c. Type III – Complex Roots: For each conjugate pair of complex roots m_1 and m_2 of the auxiliary equation, both $c_1 \cdot e^{\alpha x} \cdot \cos(\beta x)$ and $c_2 \cdot e^{\alpha x} \cdot \sin(\beta x)$ are solutions to the corresponding DE. (Where $m_1 = \alpha + \beta i$ and $m_2 = \alpha - \beta i$.) (5) The general solution y_c to the DE may be found by summing the solutions found in each part of (4), noting that the constants c_i in each solution may be different.

n^{th}-Order Linear Non-Homogenous (Annihilator Approach) (4.5)	$a_n y^{(n)} + a_{(n-1)} y^{(n-1)} + \ldots + a_0 y = g(x)$ Where $g(x) \neq 0$, a_n is a constant (possibly complex) and $y^{(n)}$ is the n[th] derivative of y, and $g(x)$ is a product or sum of polynomial, exponential, sine or cosine functions.	(1) Replace $g(x)$ with 0. Find the general solution y_c of the resulting Linear Homogenous DE. (2) Rewrite the DE using D-notation. (3) Determine an annihilation function (operator) $L(x)$ such that $L(g(x)) = 0$. a. The operator $L_1(x) = D^n$ annihilates polynomial terms of the form $c_1 \cdot x^k$, for any $k \mid 0 \leq k \leq (n-1)$ (i.e. polynomials of degree $n-1$). b. The operator $L_2(x) = (D-\alpha)^n$ annihilates exponential terms of the form $c_2 . x^k e^{\alpha x}$, for any $k \mid 0 \leq k \leq (n-1)$ c. The operator $L_3(x) = \left(D^2 - 2\alpha D - \left(\alpha^2 - \beta^2\right)\right)^n$ annihilates sine, cosine, and exponential terms of the form $c_3 . x^k e^{\alpha x} \cos(\beta x)$ and $c_3 . x^k e^{\alpha x} \sin(\beta x)$, for any $k \mid 0 \leq k \leq (n-1)$ (4) Apply the annihilation operator to both sides of the equation. Find the general solution y_p of the resulting Linear Homogenous equation. (5) Delete any terms in y_p (found in (4)) that also appear in y_c (found in (1)). Plug this reduced y_p into the original DE and solve for the constants in y_p. (6) The general solution of the original DE is $y_c + y_p$ (the y_p with the solved-for constants).

n^{th}-Order Linear Non-Homogenous (Variation of Parameters) (4.5)	$a_n y^{(n)} + a_{(n-1)} y^{(n-1)} + \ldots + a_0 y = g(x)$ Where $g(x) \neq 0$, a_n is a constant (possibly complex), $y^{(n)}$ is the n^{th} derivative of y, and $g(x)$ is any function.	(1) Replace $g(x)$ with 0. Find the general solution y_c of the resulting Linear Homogenous DE. (2) Compute the general Wronskian $W(y_{c1}, y_{c2}, \ldots, y_{cn})$, where y_{cn} is the n^{th} term of the general solution y_c with a coefficient of 1. (3) Put the original DE into standard form (i.e. divide both sides of the equation by a_n). Let $\frac{g(x)}{a_n} = f(x)$. (4) For each term y_{ci} of the general solution y_c, do the following: a. Compute the modified Wronskian W_i, which is the general Wronskian with the i^{th} column (i.e. the column tied to the i^{th} term of the general solution y_c) replaced with the function $f(x)$ in the bottom row and 0 in all the other rows. b. Compute the factor $u_i(x) = e^{\int \frac{W_i}{W} dx}$. (5) The particular solution is $y_p = \sum_{i=1}^{n} u_i y_i$ (i.e. the sum of the products of each term of y_c and their respective factors). (6) The general solution of the original DE is $y_c + y_p$.

n[th]-Order Cauchy-Euler (4.7)	$a_n x^n y^{(n)} + a_{(n-1)} x^{n-1} y^{(n-1)} + \ldots + a_0 y = g(x)$ Where $g(x) \neq 0$, a_n is a constant (possibly complex), $y^{(n)}$ is the n[th] derivative of y, and $g(x)$ is any function. (If $g(x) = 0$, then the equation is called homogenous.)	(1) Make the substitution $y = x^m$ (and $y' = mx^m$, $y'' = m(m-1)x^m$, etc.) (2) Factor out x^m from all terms. Since $e^{mx} > 0$ for all $m \in \mathbf{R}$, it can be dropped, giving a polynomial "auxiliary equation". (3) Solve the auxiliary equation through whichever means are available. Classify the roots as follows: Type I – Unique Real Roots Type II – Repeated Real Roots (i.e. those roots which bring more than one factor in the auxiliary equation to zero) Type III – Complex Roots (which always appear in conjugate pairs $\alpha + \beta i$ and $\alpha - \beta i$) (4) Transform the roots of the auxiliary equation into solutions of the DE as follows: a. Type I – Unique Real Roots: For each unique real root m of the auxiliary equation, there is a solution $c \cdot x^m$ to the corresponding DE. b. Type II – Repeated Real Roots: For each real root m of the auxiliary equation repeated k times, each of the following is a solution to the corresponding DE: $c_1 \cdot \ln\|x\|^0 \cdot x^m$, $c_2 \cdot \ln\|x\|^1 \cdot x^m$, … , $c_k \cdot \ln\|x\|^{(k-1)} \cdot x^m$ (This can be proved using reduction of order.) c. Type III – Complex Roots: For each conjugate pair of complex roots m_1 and m_2 of the auxiliary equation, both $c_1 \cdot x^\alpha \cdot \cos(\beta \cdot \ln\|x\|)$ and $c_2 \cdot x^\alpha \cdot \sin(\beta \cdot \ln\|x\|)$ are solutions to the corresponding DE. (Where $m_1 = \alpha + \beta i$ and $m_2 = \alpha - \beta i$.) (5) The general solution y_c to the DE may be found by summing the solutions found in each part of (4), noting that the constants c_i in each solution may be different. (6) If $g(x) \neq 0$ (i.e. the equation is non-homogenous), then use Variation of Parameters (using y_c from (5) and starting at step (2)) to find the particular solution y_p. (7) The general solution of the original DE is $y_c + y_p$.